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Reflection and Transmission Times Through a Linear Potential

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Reflection and transmission times through a linear potential is studied by using the stationary phase method. The incident particle is described by a wave packet constructed from a momentum distribution $\delta(k' - k)$ highly concentrated around a fixed value k. The way the reflection and transmission times are calculated is similar to the way the phase time τ_φ is defined for a rectangular potential.

KEY WORDS: reflection time; transmission time; tunneling time; linear potential.

1. INTRODUCTION

The purpose of this paper is not to dive into the turbulent water of the controversies that plague the subject of transversal or tunneling time through a barrier. Very good and comprehensive papers that properly address this issue can be found in the literature. Among them, we cite the review papers by Hauge and Støvneng (1989), Landauer and Martin (1994), and, more recently, Muga and Leavens (2000). Despite all the efforts made up to now, no consensus was achieved yet on how to define and evaluate tunneling times.

By using the stationary phase method (Erdélyi, 1956), we discuss the transmission and reflection times of a particle of mass m and energy E incident on a linear Schottky kind potential barrier of maximum intensity $V_0 > E = \hbar^2 k^2/2m$. This problem is particularly interesting to researchers in fields like heterostructures or multiple-quantum-well structures.

The particle is described by a wavepacket constructed at a distance $x = -\ell$ before the location of the potential. We take a Gaussian momentum distribution

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highly concentrated around a defined value of k. This practice reduces considerably the difficulties inherent to nonconcentrated momentum distributions.

The way to obtain the transmission and reflection times is similar to the process to obtain the phase time τ_φ in de Aquino *et al.* (1998), where a rectangular potential barrier was considered. Although there is a difference between the reflection and transmission times, due to the fact that the probability density of the faster components of the wave packet is greater than the density probability of the slower components, both converges to the phase time when the Gaussian momentum distribution tends to a delta function distribution.

By applying the same method to the linear and rectangular barriers, we can make a direct comparison of the transmission and reflection times for these two barriers.

In the next section, we present a self-contained summary of the calculations carried out in de Aquino *et al.* (1998) in order to give independence to this paper and to define notations. In Section 3, the calculations are extended to the case of a linear potential barrier. The results and discussions are presented in Section 4. Finally, the conclusions are presented in Section 5.

2. THE RECTANGULAR BARRIER

Consider a particle of energy E incident on a barrier potential $V(x)$ defined by

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & 0 < x < a \\ 0, & x > a \end{cases} \tag{2.1}$$

under the condition that in $t = 0$ the probability to find the particle is given by a Gaussian distribution centered in $x(0) = -\ell$, the distribution peak moving with the group velocity V_{g}. The time evolution of such systems is described by the wave function $\Psi(x, t)$ (de Aquino *et al.*, 1998)

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^{k_0} \phi(k)\, e^{-i(t-t_0)E/\hbar} u_k(x)\, dk \tag{2.2}$$

where $u_k(x)$ are eigenfunctions of the potential $V(x)$ and are given by

$$u_k(x) = \begin{cases} u_1(x) = A\, e^{ikx} + A'\, e^{-ikx}, & x < 0 \\ u_2(x) = B\, e^{-\rho x} + D\, e^{\rho x}, & 0 < x < a \\ u_3(x) = C\, e^{ikx}, & x > a \end{cases} \tag{2.3}$$

with

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}\, e, \qquad \rho = \sqrt{k_0^2 - k^2} \tag{2.4}$$

In Eq. (2.2), $\phi(k)$ is a Gaussian momentum distribution highly centered around some $\bar{k}$ in the interval $0 < \bar{k} < k_0$

$$\phi(k) = \frac{1}{\left(2\pi\sigma_k^2\right)^{1/4}} e^{ik\ell} e^{-(k-\bar{k})^2/4\sigma_k^2}. \tag{2.5}$$

The strategy to evaluate the reflection and transmission times consists of writing the incident (I), reflected (R), and transmitted (T) components of $\Psi(x, t)$ as

$$\psi_{\mathrm{I,R,T}}(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^{k_0} \phi_{\mathrm{I,R,T}}(k)\, e^{i f_{\mathrm{I,R,T}}(k,x,t)}\, dk \tag{2.6}$$

where the "new momentum distributions" $\phi_{\mathrm{I,R,T}}(k)$ are real functions that carry the k dependence of the constants A, A', and C of Eq. (2.3). Thus, the system time evolution can be understood by noticing that the integrals in (2.6), for a given x, give a greater contribution when the phases $f_{\mathrm{I,R,T}}(k, x, t)$ are stationary (Erdélyi, 1956), i.e., when

$$\left.\frac{d f_{\mathrm{I,R,T}}(k)}{dk}\right|_{k=\bar{k}_{\mathrm{I,R,T}}} = 0, \tag{2.7}$$

the condition in (2.7) generates the equations for the position of the peak of the incident, reflected, and transmitted wave packets, in function of time. Taking $A = 1$ in (2.3), and taking into account Eq. (2.5), we see that the $\psi_{\mathrm{I}}(x, t)$ component describes a wave packet centered around $t = t_0$ in the position $x = -\ell$, its maximum moving with the group velocity $\bar{V}_{\mathrm{g}} = +\hbar\bar{k}/m$. The time reflection τ_{R} is then obtained by finding the instant t_{R} in which the reflected wave packet "emerges" in $x = 0$ with group velocity $-\bar{V}_{\mathrm{gR}}$ and subtracting the time necessary for the wave packet peak reach the barrier

$$\tau_{\mathrm{R}} = t_{\mathrm{R}} - \frac{m\ell}{\hbar k}. \tag{2.8}$$

The transmission time can be obtained in a similar way.

These quantities result in different values for finite width momentum distributions $\phi(k)$. When the momentum distributions $\phi(k)$ tend to a delta function, the integrals in (2.6) tend to the stationary monocromatic wave functions $u_k(x)$ and the transmission and reflection time results are equal to the so-called phase time τ_φ (Hartman, 1962; Wigner, 1955)

$$\tau_\varphi = \tau_0 \frac{k_0^2 \frac{\sinh(2\rho a)}{k\rho} - \frac{2ka(k^2-\rho^2)}{k_0^2}}{(\sinh \rho a)^2 + \left(\frac{2k\rho}{k_0^2}\right)^2} \tag{2.9}$$

where

$$\tau_0 = \hbar/2V_0 \tag{2.10}$$

is the barrier characteristic time.

3. A LINEAR BARRIER

Consider a particle of energy $E < V_0$ incident, from the region of negative x, on a potential barrier $V(x)$ defined by

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0\left(1 - \frac{x}{a}\right), & 0 < x < a \\ 0, & x > a \end{cases} \tag{3.1}$$

Defining the dimensionless quantities

$$\varepsilon = \frac{k}{k_0} \quad \text{and} \quad q_0 = (k_0 a)^{2/3} \tag{3.2}$$

with k and k_0 defined in (2.4), the time-independent Schroedinger equation in the interval $0 < x < a$ can be written as

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -\left(\frac{x}{a} + \varepsilon^2 - 1\right) k_0^2\, \psi(x). \tag{3.3}$$

In terms of the convenient variable

$$\xi = \xi(x) = q_0 \left(\frac{x}{a} + \varepsilon^2 - 1\right) \tag{3.4}$$

the Eq. (3.3) reads

$$\frac{\partial^2 \psi(\xi)}{\partial \xi^2} = -\xi\, \psi(\xi), \tag{3.5}$$

whose solutions can be given in terms of Airy functions $Ai(-\xi)$ and $Bi(-\xi)$ (Abramovitz and Stegun, 1970). The general solution for a particle incident from the left on the linear barrier potential (3.1) is then given by

$$\psi_k(x) = \begin{cases} \psi_1(x) = e^{ikx} + A\, e^{-ikx}, & x < 0 \\ \psi_2(x) = B Ai(-\xi) + C Bi(-\xi), & 0 < x < a \\ \psi_3(x) = D\, e^{ikx}, & x > a. \end{cases} \tag{3.6}$$

Let us introduce the functions $F(\xi)$ and $G(\xi)$ defined by

$$F(\xi) = ik Ai(-\xi) - \frac{q_0}{a} Ai'(-\xi) \tag{3.7}$$

$$G(\xi) = ik Bi(-\xi) - \frac{q_0}{a} Bi'(-\xi) \tag{3.8}$$

where the prime means derivative with respect to the variable $-\xi$. Continuity condition of ψ_k and its spacial derivative in $x = 0$ and $x = a$ determine the constants A, B, C, and D as

$$A = A(k) = \frac{G^*(\xi_a) F^*(\xi_0) - F^*(\xi_a) G^*(\xi_0)}{\Delta} \tag{3.9}$$

$$B = B(k) = \frac{-2ikG^*(\xi_a)}{\Delta} \tag{3.10}$$

$$C = C(k) = \frac{2ikF^*(\xi_a)}{\Delta} \tag{3.11}$$

and

$$D = D(k) = \frac{F^*(\xi_a)G(\xi_a) - G^*(\xi_a)F(\xi_a)}{\Delta}\, e^{-ika} = \frac{2ikq_0}{\pi a \Delta}\, e^{-ika}, \tag{3.12}$$

where use was made of the Wronskian $W(Ai, Bi) = 1/\pi$.

In Eqs. (3.9)–(3.12),

$$\Delta = G(\xi_0)F^*(\xi_a) - F(\xi_0)G^*(\xi_a) \tag{3.13}$$

and from (3.4)

$$\begin{aligned} \xi_0 &= \xi(0) = q_0(\varepsilon^2 - 1) \\ \xi_a &= \xi(a) = q_0(\varepsilon^2) \end{aligned} \tag{3.14}$$

The asterisk in (3.9)–(3.13) means complex conjugate.

Introducing the auxiliary quantities

$$R_1 = \varepsilon^2[A_i(-\xi_0)B_i(-\xi_a) - A_i(-\xi_a)B_i(-\xi_0)] \tag{3.15}$$

$$R_2 = \frac{q_o^2}{(k_0a)^2}[A_i'(-\xi_a)B_i'(-\xi_0) - A_i'(-\xi_0)B_i'(-\xi_a)] \tag{3.16}$$

$$I_1 = \frac{\varepsilon q_o}{k_0a}[A_i(-\xi_a)B_i'(-\xi_0) - A_i'(-\xi_0)B_i(-\xi_a)] \tag{3.17}$$

$$I_2 = \frac{\varepsilon q_o}{k_0a}[A_i'(-\xi_a)B_i(-\xi_0) - A_i(-\xi_0)B_i'(-\xi_a)] \tag{3.18}$$

we can rewrite the coefficient $A(k)$ and Δ as

$$A(k) = \frac{-N_A}{\Delta} \tag{3.19}$$

and

$$\Delta = k_0^2[(R_2 - R_1) + i(I_1 - I_2)] \tag{3.20}$$

with

$$N_A = k_0^2[(R_1 + R_2) + i(I_1 + I_2)]. \tag{3.21}$$

In polar representation, N_A and Δ take the form

$$N_A = |N_A|\, e^{-\alpha} \tag{3.22}$$

$$\Delta = |\Delta|\, e^{i\lambda} \tag{3.23}$$

with

$$|N_A| = k_0^2\sqrt{(R_1+R_2)^2+(I_1+I_2)^2} \tag{3.24}$$

$$\tan\alpha = \frac{I_1+I_2}{R_1+R_2} \tag{3.25}$$

and

$$|\Delta| = k_0^2\sqrt{(R_2-R_1)^2+(I_1-I_2)^2} \tag{3.26}$$

$$\tan\lambda = \frac{I_1-I_2}{R_2-R_1} \tag{3.27}$$

In terms of these quantities, the reflected wave function reads

$$\psi_{\mathrm{R}}(x,t) = -\frac{|N_A|}{|\Delta|}\, e^{i(\alpha-\lambda)}\, e^{-ikx}\, e^{-i\omega(k)t}. \tag{3.28}$$

Imposing stationary phase condition on (3.28), in $x=0$, results

$$x + \frac{\hbar kt}{m} + \frac{d\lambda}{dk} - \frac{d\alpha}{dk}\bigg|_{x=0} = 0 \tag{3.29}$$

From (3.29), we obtain the reflection time

$$\tau_{\mathrm{R}} = \frac{m}{\hbar k}\left(\frac{d\alpha}{dk} - \frac{d\lambda}{dk}\right). \tag{3.30}$$

In terms of the characteristic time τ_0, defined in (2.10), and noticing that $\frac{\partial}{\partial k} = \frac{1}{k_0}\frac{\partial}{\partial\varepsilon}$, the reflection time τ_{R} can be written as

$$\frac{\tau_{\mathrm{R}}}{\tau_0} = \frac{1}{\varepsilon}\frac{\partial}{\partial\varepsilon}(\alpha-\lambda). \tag{3.31}$$

The derivative $\partial\alpha/\partial\varepsilon$ can be readily evaluated by noticing that

$$\frac{\partial}{\partial\alpha}(\tan\alpha)\frac{\partial\alpha}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}\left(\frac{I_1+I_2}{R_1+R_2}\right) \tag{3.32}$$

to get

$$\frac{\partial\alpha}{\partial\varepsilon} = \cos^2\alpha\,\frac{\partial}{\partial\varepsilon}\left(\frac{I_1+I_2}{R_1+R_2}\right). \tag{3.33}$$

A little more algebra produces the results

$$\frac{\partial\alpha}{\partial\varepsilon} = \frac{(R_1+R_2)\frac{\partial}{\partial\varepsilon}(I_1+I_2) - (I_1+I_2)\frac{\partial}{\partial\varepsilon}(R_1+R_2)}{|N_A|^2} \tag{3.34}$$

$$\frac{\partial\lambda}{\partial\varepsilon} = \frac{(R_2-R_1)\frac{\partial}{\partial\varepsilon}(I_1-I_2) - (I_1-I_2)\frac{\partial}{\partial\varepsilon}(R_2-R_1)}{|\Delta|^2}. \tag{3.35}$$

The derivatives of the quantities R_1, R_2, I_1, and I_2 can be obtained from known relations between the Airy functions and its derivatives (Abramovitz and Stegun, 1970). The results are

$$\frac{\partial R_1}{\partial \varepsilon} = \frac{2R_1}{\varepsilon} + 2\varepsilon^2 k_0 a(I_1 + I_2) \tag{3.36}$$

$$\frac{\partial R_2}{\partial \varepsilon} = \frac{2k_0 a}{q_0}(\xi_a I_1 + \xi_0 I_2) \tag{3.37}$$

$$\frac{\partial I_1}{\partial \varepsilon} = \frac{I_1}{\varepsilon} - 2\varepsilon^2 k_0 a R_2 - \frac{2\xi_0 k_0 a}{q_0} R_1 \tag{3.38}$$

$$\frac{\partial I_2}{\partial \varepsilon} = \frac{I_2}{\varepsilon} - 2\varepsilon^2 k_0 a R_2 - \frac{2\xi_a k_0 a}{q_0} R_1. \tag{3.39}$$

Finally, the transmission time can now be obtained by noticing that the first term of $D(k)$, Eq. (3.12), is imaginary, so that the phase f_{T} associated with the transmitted component of $\Psi(x, t)$ is given, apart a constant phase, by

$$f_{\mathrm{T}} = -ka - \lambda + kx - \omega(k)t \tag{3.40}$$

and the condition of stationary phase for f_{T} in $x = a$ leads to the result

$$\frac{\tau_{\mathrm{T}}}{\tau_0} = -\frac{1}{\varepsilon}\frac{\partial \lambda}{\partial \varepsilon}. \tag{3.41}$$

4. RESULTS AND DISCUSSION

For comparison, we show in Fig. 1 the behavior of τ_φ, Eq. (2.9), in units of τ_0, for a rectangular barrier, in function of the barrier width a, for some values of k. In this figure, we can observe the stabilization of τ_φ for very thick barriers, the

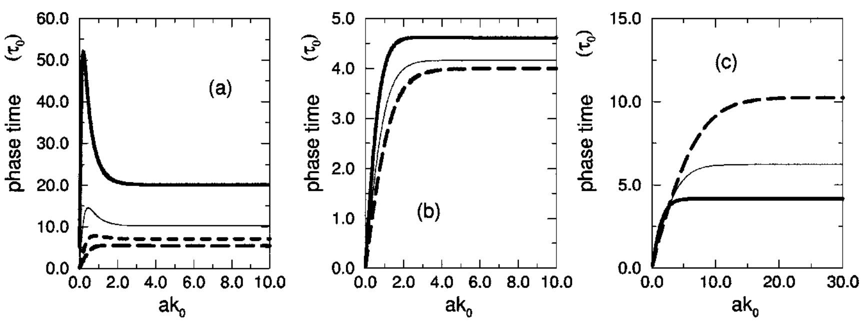

Fig. 1. Phase time τ_ϕ in units of τ_0 for a rectangular barrier. (a) Thick solid line: $k = 0.1k_0$; thin solid line: $k = 0.2k_0$; dashed line: $k = 0.3k_0$; long dashed line: $k = 0.4k_0$. (b) Thick solid line: $k = 0.5k_0$; thin solid line: $k = 0.6k_0$; dashed line: $k = 0.7k_0$. (c) Dashed line: $k = 0.98k_0$; thin solid line: $k = 0.94k_0$; thick solid line: $k = 0.8k_0$.

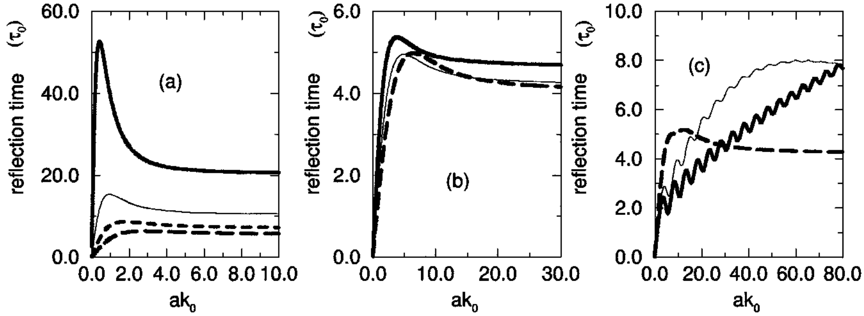

Fig. 2. Reflection time τ_R, in units of τ_0, for a linear barrier. (a) Thick solid line: $k = 0.1k_0$; thin solid line: $k = 0.2k_0$; dashed line: $k = 0.3k_0$; long dashed line: $k = 0.4k_0$. (b) Thick solid line: $k = 0.5k_0$; thin solid line: $k = 0.6k_0$; dashed line: $k = 0.7k_0$. (c) Thick solid line: $k = 0.98k_0$; thin solid line: $k = 0.94k_0$; dashed line: $k = 0.8k_0$.

so-called Hartman effect (Hartman, 1962). We also observe the existence of sharp peaks in the region of small values of k.

In Fig. 2, for the same set of values of k used in Fig. 1, we show the reflection time τ_R in units of τ_0, Eq. (3.31), for the linear barrier potential. The stabilization of τ_R for very thick barrier is not evident in this interval of values for the parameter a, for $0.5k_0 < k < k_0$, but can be observed in Fig. 3, where the oscilating behavior of τ_R with a, for $k \approx 0.94k_0$ it is not apparent due to the figure scale.

Figure 4 displays the transmission time τ_T in units of τ_0, Eq. (3.41), for the same values of k considered in the analysis of the reflection time.

5. CONCLUSION

There are some basic differences that must be noticed between the reflection time evaluated for linear and rectangular barrier potentials. Firstly, while in the

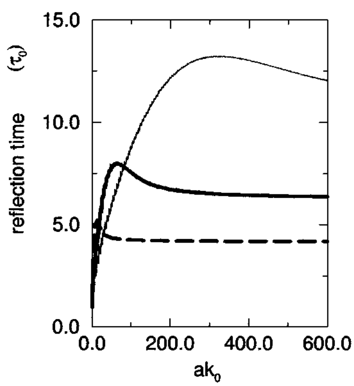

Fig. 3. Reflection time τ_R, in units of τ_0, for a linear barrier. Thin solid line: $k = 0.98k_0$; thick solid line: $k = 0.94k_0$; dashed line: $k = 0.8k_0$.

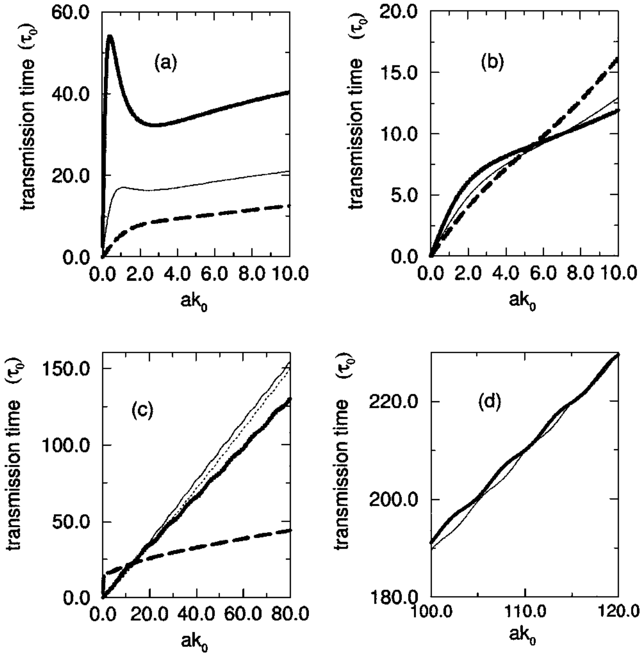

Fig. 4. Transmission time τ_T, in units of τ_0, for a linear barrier. (a) Thick solid line: $k = 0.1k_0$; thin solid line: $k = 0.2k_0$; dashed line: $k = 0.4k_0$. (b) Thick solid line: $k = 0.5k_0$; thin solid line: $k = 0.6k_0$; dashed line: $k = 0.7k_0$. (c) Thick solid line: $k = 0.8k_0$; thin dotted line: $k = 0.98k_0$; thin solid line: $k = 0.94k_0$; dashed line: $k = 0.2k_0$. (d) Thick solid line: $k = 0.94k_0$; thin solid line: $k = 0.98k_0$.

case of a rectangular barrier the stabilization of τ_φ occurs in the whole range of k, for relatively small values of the barrier thickness a, the stabilization of τ_R, for $k \gtrsim 0.8k_0$, only occurs for large values of a.

On the other hand, the existence of a maximum for the transmission time τ_R is a general characteristic in the whole range of k for linear barrier. The same is not true for rectangular barrier where τ_R presents a maximum only in the region of small k.

The second point we address is that, for both barrier cases, τ_R and τ_φ present variations of the same order for the same values of k but for different values of the barrier thickness a.

Another interesting aspect, related with the reflection time, is that the stabilization of τ_φ in the case of the linear barrier can be derived by taking the limit of large a in expression (2.9). The resulting expression gives the reflection time associated with a particle incident on a rectangular barrier of infinite thickness (step barrier) (Cohen-Tanoudji *et al.*, 1977)

$$\left(\frac{\tau_\varphi}{\tau_0}\right)_{a\to\infty} = \frac{2k_0^2}{\sqrt{k^2\left(k_0^2 - k^2\right)}}, \tag{5.1}$$

and has the smaller stabilization value $(\tau_\varphi/\tau_0)_{\min}$ for $k = \sqrt{0.5}k_0$, and is equal to 4, as can be seen directly from Eq. (5.1). A corresponding analysis of the plottings of the reflection times for a linear barrier shows that the minimum stabilization value of τ_{R}/τ_0 occurs for the same value of k, and is also equal to 4.

As a last observation on the reflection times, the quantity τ_{R} associated with the linear barrier presents an increasing oscillating behavior for $0.9k_0 < k < k_0$. This behavior is smoothened when we consider the transmission times. For a particle incident on a rectangular barrier, the transmission time τ_{T} tends to the phase time τ_φ in the limit of large width, and it does not stabilize for any values of k in the case of a linear barrier, as can be seen in Fig. 4.

REFERENCES

Abramovitz, M. and Stegun, I. (1970). *Handbook of Mathematical Functions*, Dover, New York.

Cohen-Tanoudji, C., Diu, B., and Laloë, F. (1977). *Quantum Mechanics*, *Vol. 1,* Wiley, Paris, p. 81.

de Aquino, V. M., Aguilera-Navarro, V. C., Goto, M., and Iwamoto, H. (1998). *Physical Review A* **58**, 4359.

Erdélyi, A. (1956). *Assymptotic Expansions*, Dover, New York.

Hartman, T. E. (1962). *Journal of Applied Physics* **33**, 3427.

Hauge, E. H. and Støvneng, J. A. (1989). *Review of Modern Physics* **61**, 91.

Landauer, R. and Martin, Th. (1994). *Review of Modern Physics* **66**, 217.

Muga, J. G. and Leavens, C. R. (2000). *Physics Report* **338**, 353.

Wigner, E. P. (1955). *Physical Review* **98**, 145.